Total nonnegativity of the extended Perron complement

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Abstract

A real matrix is called totally nonnegative if all of its minors are nonnegative. In this paper the extended Perron complement of a principal submatrix in a matrix A is investigated. In extension of known results it is shown that if A is irreducible and totally nonnegative and the principal submatrix consists of some specified consecutive rows then the extended Perron complement is totally nonnegative. Also inequalities between minors of the extended Perron complement and the Schur complement are presented.

Keywords: Totally nonnegative matrix, Perron complement, extended Perron complement, Schur complement.
2010 MSC: 15B48

1. Introduction

A real matrix is called totally nonnegative if all of its minors are nonnegative. For properties of these matrices the reader is referred to the two monographs [6], [10] and the survey paper [4].

In this paper we investigate the extended Perron complement of totally nonnegative matrices by using properties of determinants and determinantal inequalities as well as some results on the perturbation of tridiagonal totally nonnegative matrices [2], cf. [3]. Several interesting properties of the Perron complement of irreducible nonnegative matrices and a method to compute the Perron vector of a given irreducible nonnegative matrix by using Perron complementation and a divide-and-conquer procedure are presented in [8, 9]. Important results on the extended Perron complement of irreducible totally nonnegative matrices are given in [7].

The organization of our paper is as follows. In the next section we introduce the notation and the definitions used in the paper. We provide therein also some determinatal identities and inequalities which we employ in the proof of our main results. In Section 3 we present our main results. Here we enlarge the class of principal minors for which the extended Perron complement of an irreducible totally nonnegative matrix, A say, is known to be in turn totally nonnegative. Special emphasis is laid on the case that A is in addition tridiagonal. The paper is completed by several inequalities between the minors of the extended Perron complement, a specified complementary principal submatrix, and the Schur complement of A.

2. Notation and auxiliary results

The set of the n-by-m real matrices is denoted by $\mathbb{R}^{n,m}$ (endowed with the usual entry-wise partial ordering $\leq$). For integers κ, n, $Q_{\kappa,n}$ is the set of all strictly increasing sequences of κ integers chosen from $\{1, 2, \ldots, n\}$. We use the set theoretic symbols $\cup$ and $\setminus$ to denote somewhat not precisely but intuitively the union and the difference, respectively, of two index sequences, where we consider the resulting sequence as strictly increasing ordered. For $\alpha \in Q_{\kappa,n}$ we define $\alpha^c := \{1, \ldots, n\} \setminus \alpha$.

For $A \in \mathbb{R}^{n,m}$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\kappa) \in Q_{\kappa,n}$, and $\beta = (\beta_1, \beta_2, \ldots, \beta_\mu) \in Q_{\mu,m}$, we denote by $A[\alpha|\beta]$ the κ-by-μ submatrix of A lying in the rows indexed by $\alpha_1, \alpha_2, \ldots, \alpha_\kappa$ and columns indexed by $\beta_1, \beta_2, \ldots, \beta_\mu$. We suppress the brackets when we enumerate the indices explicitly. By $A(\alpha|\beta)$ we denote the $(n-\kappa)$-by-$(m-\mu)$ submatrix $A[\alpha^c|\beta^c]$ of A. When $\alpha = \beta$, the *principal submatrix* $A[\alpha|\alpha]$ is abbreviated to $A[\alpha]$ and $\det A[\alpha]$ is called a *principal minor*, with the similar notation $A(\alpha)$ for the complementary principal submatrix. We also introduce the following notations which simplify the presentation. For $\alpha = \{\alpha_1, \ldots, \alpha_\kappa\} \in Q_{\kappa,n-1}$ put

$$\alpha + 1 := \{\alpha_1 + 1, \ldots, \alpha_\kappa + 1\}, \quad \alpha_{\hat{1}} + 1 := \{\alpha_1, \alpha_2 + 1, \ldots, \alpha_\kappa + 1\}.$$

A matrix $A = (a_{ij}) \in \mathbb{R}^{n,n}$ is referred to as *tridiagonal* (or a *Jacobi matrix*) if $a_{ij} = 0$ whenever $|i-j| > 1$, it is termed *nonnegative* if $A \geq 0$, and it is called *totally nonnegative* (abbreviated TN) if $\det A[\alpha|\beta] \geq 0$, for all $\alpha, \beta \in Q_{\kappa,n}$. If A is TN and in addition nonsingular we write A is $NsTN$. In passing, we note that if A is TN then so are its transpose and $A^{\#} := T_n A T_n$, where $T_n = (t_{ij})$ is the permutation matrix of order n with $t_{ij} = \delta_{i,n-j+1}$, $i, j = 1, \ldots, n$, see, e.g., [6, Theorem 1.4.1].

A matrix $A \in \mathbb{R}^{n,n}$ is termed *irreducible* if either $n = 1$ and $A \neq 0$ or $n \geq 2$ and there is no permutation matrix P such that

$$PAP^T = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix},$$

where 0 is the $(n-r)$-by-r zero matrix $(1 \leq r \leq n-1)$. Otherwise it is called *reducible*.

For a matrix $A \in \mathbb{R}^{n,n}$ with $A[\alpha]$ is nonsingular for some $\alpha \in Q_{\kappa,n}$, the *Schur complement of* $A[\alpha]$ *in* A, denoted by $A/A[\alpha]$, is defined as

$$A/A[\alpha] := A[\alpha^c] - A[\alpha^c|\alpha](A[\alpha])^{-1}A[\alpha|\alpha^c]. \tag{1}$$

The following theorem is important for the definition of the (extended) Perron complement of an irreducible nonnegative matrix.

Theorem 2.1. *[5, Corollary (1.5), p.27] Let $A, B \in \mathbb{R}^{n,n}$ be such that $0 \leq B \leq A$ and $A + B$ is irreducible. Then $\rho(B) < \rho(A)$, where $\rho(\cdot)$ denotes the* spectral radius.

Definition 2.1. *[6, Section 10.4] Let $A \in \mathbb{R}^{n,n}$ be an irreducible nonnegative matrix. Then the* Perron complement of $A[\alpha]$ in A *is given by*

$$\mathcal{P}(A/A[\alpha]) := A[\beta] + A[\beta|\alpha](\rho(A)I - A[\alpha])^{-1}A[\alpha|\beta], \tag{2}$$

where $\alpha \subset \{1, \ldots, n\}$ and $\beta := \alpha^c$.

Remark 2.1. *The expression on the right-hand side of* (2) *is well defined by Theorem 2.1 since A is irreducible and nonnegative and hence $\rho(A[\alpha]) < \rho(A)$.*

The Perron complement is extended in the following way [7]:
For any $\rho(A) \leq t$, α and β as in Definition 2.1 define

$$\mathcal{P}_t(A/A[\alpha]) := A[\beta] + A[\beta|\alpha](tI - A[\alpha])^{-1}A[\alpha|\beta], \tag{3}$$

which is called the *extended Perron complement of* $A[\alpha]$ *in* A *at* t. Again $\mathcal{P}_t(A/A[\alpha])$ is well defined, see Remark 2.1.

There is an interesting relationship between Sylvester's determinantal identity, see, e.g., [6, p.29], and the Schur complement given by (1) for an n-by-n matrix A [4, p.175], see also [6, formula (10.4)]. Let $\alpha = \{k, \ldots, n\}$ or $\alpha = \{1, \ldots, \kappa\}$. Then

$$\det(A/A[\alpha])[\gamma|\delta] = \frac{\det A[\gamma \cup \alpha|\delta \cup \alpha]}{\det A[\alpha]}, \tag{4}$$

where $2 \leq k \leq n$, $1 \leq \kappa \leq n-1$, and $\gamma, \delta \subseteq \alpha^c$ with $|\gamma| = |\delta|$.

From the above equalities we have the following theorem.

Theorem 2.2. *[4, Theorem 3.7] Let $A \in \mathbb{R}^{n,n}$ be TN and $\alpha = \{1, \ldots, k\}$ or $\alpha = \{k, \ldots, n\}$. Then $A/A[\alpha]$ is TN for all $k = 1, \ldots, n-1$, provided that $A[\alpha]$ is nonsingular.*

The following theorem and two lemmata play an important role in showing the total nonnegativity and nonsingularity of the extended perron complement.

Theorem 2.3. *[6, Corollary 6.2.4, Koteljanskiĭ Inequality] Let $A \in \mathbb{R}^{n,n}$ be TN. Then for any $\alpha \in Q_{\kappa,n}$ and $\beta \in Q_{\mu,n}$, the following inequality holds:*

$$\det A[\alpha \cup \beta] \cdot \det A[\alpha \cap \beta] \leq \det A[\alpha] \cdot \det A[\beta], \tag{5}$$

with the convention $\det A[\phi] := 1$.

Lemma 2.1. *E.g., [10, Theorem 1.13] All principal minors of a $NsTN$ matrix are positive.*

Lemma 2.2. *[2, Lemma 12] Let $A \in \mathbb{R}^{n,n}$ be an irreducible, tridiagonal TN matrix. Then* $\det A(i) > 0$, $i = 1, \ldots, n$.

3. Main results

In this section we present our main results. In [7, p.90], see also [6, Example 10.4.3], Fallat and Neumann give an example with $n = 10$, $\alpha = \{7\}$ which documents that TN matrices are not closed under arbitrary Perron complementation, even when α is a singleton, except $\alpha = \{1\}$ or $\alpha = \{n\}$. For the following theorem we present a new proof because we will extend this theorem by similar means, see Theorem 3.2.

Theorem 3.1. *[7, Lemma 2.2], [6, Lemma 10.4.2] Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be irreducible TN, and $\alpha = \{1\}$ or $\alpha = \{n\}$. Then for any t, $\rho(A) \leq t$, the matrix $\mathcal{P}_t(A/A[\alpha])$ is TN.*

Proof. We give the proof only for the case $\alpha = \{1\}$ since the other case follows by application of the same arguments to $A^{\#}$. Formula (3) specifies for $\alpha = \{1\}$ to

$$\mathcal{P}_t(A/A[1]) = A[2,\dots,n] + \frac{1}{t - a_{11}} A[2,\dots,n|1]A[1|2,\dots,n]. \tag{6}$$

By direct computations, it is easy to see that

$$\mathcal{P}_t(A/A[1])[1,\dots,n-1|j] = A[2,\dots,n|j+1] + \frac{a_{1,j+1}}{t - a_{11}} A[2,\dots,n|1], \tag{7}$$

for $j = 1,\dots,n-1$.
For any $\gamma = \{\gamma_1,\dots,\gamma_l\}, \delta = \{\delta_1,\dots,\delta_l\} \in Q_{l,n-1}$, $l = 1,\dots,n-1$, it is advantageous to represent $\det \mathcal{P}_t(A/A[1])[\gamma|\delta]$ in the following way

$$\det \mathcal{P}_t(A/A[1])[\gamma|\delta] = \det \begin{bmatrix} 1 & 0 \\ A[\gamma+1|1] & \mathcal{P}_t(A/A[1])[\gamma|\delta] \end{bmatrix}. \tag{8}$$

Then we subtract in the matrix on the right-hand side of (8) from the μ^{th} column the first column multiplied by $\frac{a_{1,\delta_{\mu-1}+1}}{t-a_{11}}$, $\mu = 2,\dots,l+1$, and extract from the first row the common factor $\frac{1}{t-a_{11}}$ to obtain

$$\begin{aligned}
&\det \mathcal{P}_t(A/A[1])[\gamma|\delta] \\
&\quad = \frac{1}{t-a_{11}} \det \begin{bmatrix} t - a_{11} & -A[1|\delta+1] \\ A[\gamma+1|1] & A[\gamma+1|\delta+1] \end{bmatrix} \\
&\quad = \frac{1}{t-a_{11}} (t \det A[\gamma+1|\delta+1] - \det A[\{1\} \cup (\gamma+1)|\, \{1\} \cup (\delta+1)]) \\
&\quad \geq \frac{1}{t-a_{11}} (t \det A[\gamma+1|\delta+1] - a_{11} \det A[\gamma+1|\delta+1]) && (9) \\
&\quad = \det A[\gamma+1|\delta+1], && (10)
\end{aligned}$$

where inequality (9) follows by using Theorem 2.3. Hence all minors of $\mathcal{P}_t(A/A[1])$ are nonnegative and so $\mathcal{P}_t(A/A[1])$ is TN. $\square$

In the next theorem we extend the above statement to two further special singleton sets.

Theorem 3.2. *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be irreducible TN, and $\alpha = \{2\}$ or $\alpha = \{n-1\}$. Then for any t, $\rho(A) \leq t$, the matrix $\mathcal{P}_t(A/A[\alpha])$ is TN.*

Proof. Again we provide the proof only for the case $\alpha = \{2\}$ since the other case follows by application of the same arguments to $A^{\#}$. Formula (3) specifies for $\alpha = \{2\}$ to

$$\mathcal{P}_t(A/A[2]) = A(2) + \frac{1}{t - a_{22}} A[\{2\}^c\,|2]A[2|\,\{2\}^c]. \tag{11}$$

As in the proof of Theorem 3.1, we have

$$\mathcal{P}_t(A/A[2])[1,\ldots,n-1|j] = \begin{cases} A[\{2\}^c\,|1] + \frac{a_{21}}{t-a_{22}} A[\{2\}^c\,|2], & j = 1, \\ \\ A[\{2\}^c\,|j+1] + \frac{a_{2,j+1}}{t-a_{22}} A[\{2\}^c\,|2], & j = 2,\ldots,n-1. \end{cases}$$

For any $\gamma = \{\gamma_1, \ldots, \gamma_l\}, \delta = \{\delta_1, \ldots, \delta_l\} \in Q_{l,n-1}$, $l = 1, \ldots, n-1$, we distinguish the following four cases. The equalities in the first two cases follow by using properties of determinants as in the proof of Theorem 3.1 and the inequalities follow by using Theorem 2.3.

(i) If $1 \in \gamma \cap \delta$, then

$$\begin{aligned} \det \mathcal{P}_t(A/A[2])[\gamma|\delta] &= \frac{1}{t-a_{22}}(t \det A[\gamma_{\hat{1}} + 1|\delta_{\hat{1}} + 1] \\ &\quad - \det A[\{2\} \cup (\gamma_{\hat{1}} + 1)|\,\{2\} \cup (\delta_{\hat{1}} + 1)]) \\ &\geq \det A[\gamma_{\hat{1}} + 1|\delta_{\hat{1}} + 1]. \end{aligned} \tag{12}$$

(ii) If $1 \notin \gamma \cup \delta$, then

$$\begin{aligned} \det \mathcal{P}_t(A/A[2])[\gamma|\delta] &= \frac{1}{t-a_{22}}(t \det A[\gamma + 1|\delta + 1] \\ &\quad - \det A[\{2\} \cup (\gamma + 1)|\,\{2\} \cup (\delta + 1)]) \\ &\geq \det A[\gamma + 1|\delta + 1]. \end{aligned}$$

(iii) If $1 \in \gamma$ and $1 \notin \delta$, then

$$\begin{aligned} \det \mathcal{P}_t(A/A[2])[\gamma|\delta] &= \frac{1}{t-a_{22}}(t \det A[\gamma_{\hat{1}} + 1|\delta + 1] \\ &\quad + \det A[\{2\} \cup (\gamma_{\hat{1}} + 1)|\,\{2\} \cup (\delta + 1)]) \\ &\geq 0, \end{aligned}$$

since A is TN.

(iv) If $1 \notin \gamma$ and $1 \in \delta$, then

$$\begin{aligned}\det \mathcal{P}_t(A/A[2])[\gamma|\delta] &= \frac{1}{t-a_{22}}(t \det A[\gamma+1|\delta_{\hat{1}}+1] \\ &\quad + \det A[\{2\} \cup (\gamma+1)|\,\{2\} \cup (\delta_{\hat{1}}+1)]) \\ &\geq 0,\end{aligned}$$

since A is TN.

Hence all minors of $\mathcal{P}_t(A/A[2])$ are nonnegative and so $\mathcal{P}_t(A/A[2])$ is TN. $\square$

Remark 3.1. *By an easy and direct proof one can show that $\mathcal{P}_t(A/A[\alpha])$ in Theorems 3.1 and 3.2 is irreducible. If, in addition, the given matrix in these theorems is nonsingular, then by Lemma 2.1,* (10), *and* (12) *the extended Perron complement is also nonsingular.*

Unfortunately, the above two theorems cannot be extended to any singleton set $\{k\}$, $3 \leq k \leq n-2$ with $5 \leq n$ as the following remark and example demonstrate.

Remark 3.2. *There are infinitely many totally nonnegative matrices $A = (a_{ij}) \in \mathbb{R}^{5,5}$ such that $\mathcal{P}_t(A/A[3])$ is not TN. For instance, consider the matrix A for which the only zero minor is* $\det A[1,2|4,5]$. *Such matrices can be easily found by using the so-called the Restoration Algorithm, see e.g., [1, p.42]. By direct calculations, one obtains that*

$$\begin{aligned}\det \mathcal{P}_t(A/A[3])[1,2|3,4] &= \frac{1}{t-a_{33}}(t \det A[1,2|4,5] - \det A[1,2,3|3,4,5]) \\ &= \frac{-\det A[1,2,3|3,4,5]}{t-a_{33}} < 0.\end{aligned}$$

For instance, consider the following illustrative example.

Example 3.1. *[1, Example 5.3] Let*

$$A := \begin{bmatrix} 50 & 25 & 11 & 4 & 1 \\ 35 & 20 & 10 & 4 & 1 \\ 15 & 10 & 6 & 3 & 1 \\ 5 & 4 & 3 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}. \tag{13}$$

Then A is irreducible TN,

$$\mathcal{P}_t(A/A[3]) = \begin{bmatrix} 50+\frac{165}{t-6} & 25+\frac{110}{t-6} & 4+\frac{33}{t-6} & 1+\frac{11}{t-6} \\ 35+\frac{150}{t-6} & 20+\frac{100}{t-6} & 4+\frac{30}{t-6} & 1+\frac{10}{t-6} \\ 5+\frac{45}{t-6} & 4+\frac{30}{t-6} & 2+\frac{9}{t-6} & 1+\frac{3}{t-6} \\ 1+\frac{15}{t-6} & 1+\frac{10}{t-6} & 1+\frac{3}{t-6} & 1+\frac{1}{t-6} \end{bmatrix}, \tag{14}$$

and $\det \mathcal{P}_t(A/A[3])[1,2|3,4] = \frac{-1}{t-6} < 0$ *for any* t, $\rho(A) \approx 72.756 \leq t$.

The following theorem provides a quotient formula for the extended Perron complement.

Theorem 3.3. *[7, Theorem 2.4], [6, Theorem 10.4.4] Let $A \in \mathbb{R}^{n,n}$ be an irreducible nonnegative matrix, and fix any nonempty set $\alpha \subset \{1, \ldots, n\}$. Then for any nonempty subsets $\alpha_1, \alpha_2 \subset \alpha$ with $\alpha_1 \cup \alpha_2 = \alpha$ and $\alpha_1 \cap \alpha_2 = \phi$, we have*

$$\mathcal{P}_t(A/A[\alpha]) = \mathcal{P}_t(\mathcal{P}_t(A/A[\alpha_1])/\mathcal{P}_t(A/A[\alpha_1])[\alpha_2]) \tag{15}$$

for any t, $\rho(A) \leq t$.

By using Theorems 3.1, 3.2, and 3.3, we obtain the following theorem.

Theorem 3.4. *Let $A \in \mathbb{R}^{n,n}$ be irreducible TN, and α be any one of the following subsets:*

(i) $\{1, \ldots, k\}$ *or* $\{k, \ldots, n\}$;

(ii) $\{2, \ldots, k\}$ *or* $\{k, \ldots, n-1\}$,

for $k = 2, \ldots, n-1$. Then for any t, $\rho(A) \leq t$, the matrix $\mathcal{P}_t(A/A[\alpha])$ is TN.

The case (i) in the above theorem is given in [7, Theorem 2.5], see also [6, Theorem 10.4.5].

Theorems 3.1 and 3.2 can be extended to any singleton $\{k\}$ for the class of irreducible tridiagonal TN matrices as the following theorem documents. It is a special case of Theorem 3.6 below; it was first proved in [7, Proposition 2.1]; the following proof and the statement on the nonsingularity of $\mathcal{P}_t(A/A[k])$ are new.

Theorem 3.5. *Let $A = (a_{ij}) \in \mathbb{R}^{n,n}$ be irreducible tridiagonal TN. Then for any singleton subset $\alpha = \{k\}$, $k = 1, \ldots, n$, the extended Perron complement $\mathcal{P}_t(A/A[k])$ is irreducible tridiagonal $NsTN$ for any t, $\rho(A) \leq t$.*

Proof. For $k = 1, 2, n-1, n$ the total nonnegativity of $\mathcal{P}_t(A/A[k])$ follows by Theorems 3.1 and 3.2, whereas the nonsingularity follows by (10), (12), and Lemma 2.2. Suppose that $2 < k < n-1$. Then formula (3) specifies to

$$\mathcal{P}_t(A/A[k]) \;\; = \;\; A(k) + \frac{1}{t - a_{kk}} A[\{k\}^c\,|k]A[k|\,\{k\}^c]. \tag{16}$$

By direct computations, $\mathcal{P}_t(A/A[k])$ is an irreducible and tridiagonal matrix which coincides with $A(k)$ except in the following four positions:

(i) $(k-1, k-1)$, (k, k) which become $a_{k-1,k-1} + \frac{1}{t-a_{kk}} a_{k-1,k} \cdot a_{k,k-1}$, $a_{k+1,k+1} + \frac{1}{t-a_{kk}} a_{k+1,k} \cdot a_{k,k+1}$, respectively;

(ii) $(k, k-1)$ which becomes $\frac{1}{t-a_{kk}} a_{k+1,k} \cdot a_{k,k-1}$;

(ii) $(k-1, k)$ which becomes $\frac{1}{t-a_{kk}} a_{k-1,k} \cdot a_{k,k+1}$.

We consider $A(k)$ and add first to the diagonal entries the positive quantities that appear in (i). By [4, Corollary 2.4] and Lemma 2.2 the resulting matrix is $NsTN$. Next we add the quantity in (ii) to the position $(k, k-1)$. Since the resulting matrix, called B, has at position $(k-1, k)$ a zero entry it is $NsTN$ by [2, Theorem 9]. It remains to add the quantity that appears in (iii) to the zero position $(k-1, k)$. By [2, Theorem 9] the resulting matrix is $NsTN$ if

$$\frac{1}{t - a_{kk}} a_{k-1,k} \cdot a_{k,k+1} < \frac{\det B}{\det B(k-1|k)}. \tag{17}$$

By [10, formula (4.1)] and some simplifications, the right-hand side of (17) becomes

$$\frac{(t - a_{kk}) \det A[1, \ldots, k-1] + a_{k-1,k} a_{k,k-1} \det A[1, \ldots, k-2]}{a_{k,k-1} a_{k+1,k} \det A[1, \ldots, k-2] \det A[k+2, \ldots, n]}$$
$$\cdot (\det A[k+1, \ldots, n] + \frac{1}{t - a_{kk}} a_{k,k+1} a_{k+1,k} \det A[k+2, \ldots, n]),$$

which is a sum consisting of positive terms (by irreducibility of A and Lemma 2.2) and

$$\frac{1}{t - a_{kk}} a_{k-1,k} \cdot a_{k,k+1},$$

whence inequality (17) holds. Hence $\mathcal{P}_t(A/A[k])$ is $NsTN$ for all $k = 1, \ldots, n$ and $\rho(A) \leq t$. $\square$

Theorem 3.6. *[7, Corollary 2.6], [6, Corollary 10.4.6] Let $A \in \mathbb{R}^{n,n}$ be irreducible tridiagonal TN. Then for any $\alpha \subset \{1, \ldots, n\}$, the extended Perron complement is irreducible tridiagonal TN for any t, $\rho(A) \leq t$.*

By using Theorems 3.3, 3.5, and Remark 3.1 we may conclude that under the conditions of Theorem 3.6 the resulting extended Perron complement is, in addition, nonsingular.

We conclude the paper with two theorems which compare for a TN matrix A corresponding minors of $\mathcal{P}_t(A/A[\alpha])$, $A(\alpha)$, and $A/A[\alpha]$. For simplicity of notation we put $B_1 := \mathcal{P}_t(A/A[\alpha])$, $B_2 := A(\alpha)$, and $B_3 := A/A[\alpha]$ for any $\rho(A) \leq t$ and $\alpha \in Q_{\kappa,n}$ and assume that the indexing of any complement (Perron and Schur) and accordingly of $A(\alpha)$ is inherited from the indexing of the original matrix. For example, if $A \in \mathbb{R}^{10,10}$, and $\alpha = \{2, 3, 4, 7\}$, then the rows and columns of B_1, B_2, and B_3 are indexed by the integers $1, 5, 6, 8, 9, 10$ (ordered).

Theorem 3.7. *Let $A \in \mathbb{R}^{n,n}$ be irreducible TN, and $\alpha \in \{\{1\}, \{2\}, \{n-1\}, \{n\}\}$. Then for any $\gamma, \delta \in Q_{l,n}$, $l = 1, \ldots, n-1$, the following inequalities hold:*

$$\det B_1[\gamma|\delta] \geq \det B_2[\gamma|\delta] \geq \det B_3[\gamma|\delta]. \tag{18}$$

Proof. We present the proof only for the cases $\alpha = \{1\}$ and $\alpha = \{2\}$ since the other two cases follow analogously.
Case 1. $\alpha = \{1\}$.
By (10) we have $\det B_1[\gamma|\delta] \geq \det B_2[\gamma|\delta]$ and by (4) $\det B_3[\gamma|\delta] = \frac{\det A[\gamma \cup \{1\}|\delta \cup \{1\}]}{a_{11}}$, whence by (5) $\det B_2[\gamma|\delta] \geq \det B_3[\gamma|\delta]$.
Case 2. $\alpha = \{2\}$.
By (i)-(iv) in the proof of Theorem 3.2 we have $\det B_1[\gamma|\delta] \geq \det B_2[\gamma|\delta]$. If $1 \in \gamma \cap \delta$ or $1 \notin \gamma \cup \delta$, then by Sylvester's determinantal identity [6, p.29] and (5) we have $\det B_2[\gamma|\delta] \geq \det B_3[\gamma|\delta]$ and if $1 \in \gamma \cup \delta$ and $1 \notin \gamma \cap \delta$, then $\det B_3[\gamma|\delta] \leq 0$ and hence $\det B_2[\gamma|\delta] \geq \det B_3[\gamma|\delta]$ since A is TN. $\square$

The cases $\alpha = \{1\}$ or $\alpha = \{n\}$ in the above theorem and Statement 1. in the following theorem are contained in [7, Theorem 2.7]. The next theorem provides in (ii) an additional case.

Theorem 3.8. *Let $A \in \mathbb{R}^{n,n}$ be irreducible TN. Then the following two statements hold:*

(i) If $\alpha = \{1, \dots, k_1\}$ for some $k_1 < n$, then for any $\gamma, \delta \in Q_{l,n}$, $l = 1, \dots, n - k_1$, the following inequalities hold:

$$\det B_1[\gamma|\delta] \geq \det B_2[\gamma|\delta] \geq \det B_3[\gamma|\delta], \tag{19}$$

(ii) If $\alpha = \{2, \dots, k_2\}$ for some $1 < k_2 < n$, then for any $\gamma, \delta \in Q_{l,n}$, $l = 1, \dots, n - k_2$, the following inequality holds:

$$\det B_1[\gamma|\delta] \geq \det B_2[\gamma|\delta], \tag{20}$$

and if in addition $1 \notin \gamma \cup \delta$ the following inequality holds:

$$\det B_2[\gamma|\delta] \geq \det B_3[\gamma|\delta]. \tag{21}$$

Proof. (i) Let $\alpha = \{1, \dots, k_1\}$. Then the left inequality of (19) follows by using Theorems 3.7 and 3.3 and the other inequality follows by using (4) and (5).
(ii) Let $\alpha = \{2, \dots, k_2\}$. Then (20) is again a consequence of Theorems 3.7 and 3.3. For (21), it is easy to see by using (4) that for any $\gamma, \delta \in Q_{l,n-k_2}$, $l = 1, \dots, n - k_2$, with $1 \notin \gamma \cup \delta$ the equality

$$\det B_2[\gamma|\delta] = \frac{\det A[\gamma \cup \alpha|\delta \cup \alpha]}{\det A[\alpha]}$$

holds. Hence by (5) the inequality (21) follows. □

Acknowledgements

The first author would like to thank Professor Shaun Fallat for reviewing his dissertation [1] (supervised by the second author) of which this paper has grown out.

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